# Risk Budgets

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# Risk Budgets

*Converting mean-variance optimization into VaR assignments.*

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Value at risk is generally understood to describe the maximum loss an investment could incur at a given confidence level over a specified horizon. The concept was introduced a few years ago as an innovation in risk measurement, but in fact it is a simple rearrangement of a commonly used metric of risk exposure. For more than 25 years investment managers have evaluated portfolios according to their likelihood of loss.

Suppose, for example, we estimate a portfolio's expected return and standard deviation to equal 7.10% and 18.20%, and we assume that returns are lognormally distributed.[1] It is straightforward to estimate the probability that this portfolio will suffer a loss of at least 20.00% in any given year. We begin by converting the periodic expected return and standard deviation into their continuous counterparts, as shown.

$$\mu_C = \ln (1 + \mu_P) - \sigma_C^2/2 \quad (1)$$

$$\sigma_C = [\ln (\sigma_P^2/(1+\mu_P)^2 + 1)]^{1/2} \quad (2)$$

where:

$\mu_C$ = continuous expected return;
$\mu_P$ = periodic expected return;
$\sigma_C$ = standard deviation of continuous returns;
$\sigma_P$ = standard deviation of periodic returns.

The continuous expected return and standard deviation equal 5.44% and 16.87%, respectively. Next we convert –20.00% to its continuous counterpart, which equals –22.31% [ln (0.80)], and we calculate the area to the left of –22.31% under the normal distribution. We do so by first dividing the distance between –22.31% and 5.44% by

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the continuous standard deviation, 16.87%, which equals 1.645. This value is called the normal deviate, and it means that –22.31% is 1.645 standard deviation units below the continuous expected return of 5.44%.

When we look this value up in a normal distribution table, we find that 5.00% of the area under a normal distribution with a 5.44% mean and a 16.87% standard deviation is to the left of –22.31%. Therefore, the likelihood of a 20% loss equals 5.00%, because –20.00% is the periodic counterpart of a –22.31% continuous return ($e^{-0.2231} - 1 = -0.20$).

Value at risk turns the question around. Instead of measuring the likelihood of a prespecified loss, value at risk measures how much can be lost given a prespecified likelihood. Because we know that a 5.00% probability corresponds to a normal deviate of 1.645, we estimate the amount that can be lost with a 5.00% probability as 5.44% − (1.645 × 16.87%), which equals –22.31% and which corresponds to a 20.00% periodic loss.

Actually, value at risk goes one step farther, and converts this percentage loss to a monetary value. Thus the value at risk of a $100 million portfolio with a 7.10% expected return and 18.20% standard deviation equals $20 million. The relation between probability of loss and value at risk is calculated as follows.

Probability of Loss

$$(\ln(1 + L) - \mu_C)/\sigma_C = Z \tag{3}$$

$$N(Z) = P \tag{4}$$

Value at Risk

$$\mu_C - Z\,\sigma_C = L_C \tag{5}$$

$$-(e^{L_C} - 1)\,V = VaR \tag{6}$$

where the new variables are:

$L$ = percentage loss in periodic units;
$Z$ = normal deviate;
$L_C$ = percentage loss in continuous units;
$N(\,)$ = normal distribution function;
$P$ = probability of loss;
$V$ = portfolio value; and
$VaR$ = value at risk.

To be fair, we have described the simplest conception of value at risk. Portfolio returns are not always lognormally distributed. The inclusion of derivatives or the application of dynamic trading rules may introduce additional skewness to a portfolio's return distribution, and illiquidity and trading interruptions may increase the likelihood of extreme returns, which would cause a portfolio's return distribution to have fatter tails than a lognormal distribution. Under these conditions, it is prudent to employ bootstrapping techniques to estimate value at risk.[3]

Why has value at risk captured so much attention in recent years, when in fact the investment community has been applying a variation of it for more than a quarter of a century? Its popularity arose in response to the highly publicized financial disasters that have plagued the derivatives industry, exemplified in Barings, Orange County, and Metallgesellschaft. In anticipation of legislative and regulatory intervention, the private sector introduced risk measurement systems that include value at risk calculations. And, as expected, regulators including the Office of the Comptroller of the Currency, the Financial Accounting Standards Board, and the Securities and Exchange Commission all began to scrutinize risk management practices within the derivatives industry.

The strongest impetus for the use of value at risk, however, came from bank regulators. The Group of 10, meeting in Basle, Switzerland, in 1988 agreed to establish uniform capital adequacy requirements for commercial banks to guard against credit risk. By 1993, this initial agreement evolved into an explicit proposal to use a standardized value at risk model for measuring exposure to fluctuations in interest rates, exchange rates, equity prices, and commodity prices. Finally, by 1995, the Basle Committee endorsed the banking industry's use of proprietary value at risk models.[4]

Although financial institutions typically calculate value at risk over short horizons such as a single day or a week, investment managers have begun to use value at risk to gauge their portfolios' exposure to risk over longer horizons such as a month, a quarter, or a year, which brings us to the topic of risk budgets. See Jorion [1997] for more on value at risk.

## RISK BUDGETS

We have queried a variety of people in both academic and practitioner communities on their perception of a risk budget. Two definitions seem to prevail.

### Efficient Portfolio Allocations

Some people define a risk budget as a plan for converting a portfolio's monetary allocations to various categories into value at risk assignments. These categories can be defined in a variety of ways such as asset classes, investment strategies, investment managers, or risk factors. This perception of a risk budget is appropriate as long as it follows from mean-variance optimization. It would be inefficient, however, to plan the allocation of value at risk independently of mean-variance optimization, assuming returns are lognormally distributed. Any portfolio that is efficient with respect to value at risk must lie on the mean-variance efficient frontier.

The intuition here is that, for any given expected return, a portfolio located on the efficient frontier has the lowest standard deviation. Thus, for any given expected return, the portfolio with the lowest value at risk must also lie on the efficient frontier. It follows, therefore, that any risk budget that is not mean-variance efficient has a higher portfolio value at risk for a given confidence level than a corresponding portfolio that is mean-variance efficient.

There is another reason we should not regard risk budgeting as a process for determining efficient portfolio allocations, but rather as a means for converting efficient portfolio allocations into value at risk assignments. Portfolio choice based on minimizing value at risk implies an improbable attitude toward risk. It would look with infinite opprobrium at any breach of value at risk, regardless of the extent, and it would give no credit to outcomes that offer a substantial cushion with respect to value at risk. Very few investors have such a narrow attitude toward risk.

Instead, investors should consider portfolios all along the efficient frontier and evaluate them on the basis of their entire probability distributions. By employing this approach to portfolio selection, an investor implicitly considers a wide range of values at risk and therefore many possible outcomes, not just a single threshold.

We should regard a risk budget as an extension of mean-variance optimization that enables us to decouple a portfolio's allocations from fixed monetary values. Suppose, for example, the optimal allocation of a $100 million fund calls for the percentage allocations shown in Exhibit 1. These allocations are optimal for an investor whose risk aversion equals 2.5, given the assumptions about expected returns, standard deviations, and correlations in Exhibit 1.

The traditional approach for implementing these portfolio allocations would be to invest $20.70 million in asset 1, $30.73 million in asset 2, and $48.57 million in asset 3. The risk budget alternative would instead view these allocations as value at risk assignments of $4.9567 million for asset 1, $4.5429 million for asset 2, and $4.6663 million for asset 3, which are the respective amounts that each assignment could lose with a 5% probability over a one-year horizon.

Note that the sum of the individual values at risk is nearly twice as large as the portfolio value at risk ($14.1659 million versus $7.1453 million). The reason for the discrepancy is that the assets are less than perfectly correlated with each other and therefore introduce diversification to the portfolio. Also note that although the portfolio has the least exposure to asset 1 and the greatest exposure to asset 3, asset 1 has a higher value at risk than asset 3.

This interpretation of a portfolio's allocation offers the flexibility to allocate a smaller monetary amount to each asset and leverage it. This leveraged investment would contribute the same marginal utility to the portfolio, which is tantamount to preserving the portfolio's optimality, if two conditions prevail:

- The leveraged investment preserves the expected return, volatility, and correlation with the balance of the portfolio as assumed by the original percentage allocation.
- The balance of the portfolio preserves the expected return and risk attributes assumed in the original optimization.

The economic equivalence of a value at risk assignment and a monetary allocation will prevail for any confidence level used to measure value at risk as long as returns are lognormally distributed. Thus a risk budget, in effect, simply maps a portfolio's percentage allocations

**EXHIBIT 1**
**RISK BUDGET**

| Asset | Periodic Expected Return (%) | Periodic Standard Deviation (%) | Continuous Expected Return (%) | Continuous Standard Deviation (%) | Correlations (%) Asset 1 | Asset 2 | Asset 3 | Percentage Allocation (%) | Value at Risk (per $100) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 12.00 | 25.00 | 8.90 | 22.05 | 100 | 30 | 10 | 20.70 | 4.9567 |
| 2 | 8.00 | 15.00 | 6.74 | 13.82 | 30 | 100 | 5 | 30.73 | 4.5429 |
| 3 | 6.00 | 10.00 | 5.38 | 9.41 | 10 | 5 | 100 | 48.57 | 4.6663 |
| Portfolio | 7.80 | 9.54 | 7.12 | 8.84 | | | | 100.00 | 7.1453 |

onto value at risk assignments, which offers the benefit of freeing portfolio allocations from monetary constraints—but a risk budget is efficient only if it is determined by mean-variance optimization.

### Sensitivities to Changes in Exposures

The second definition of a risk budget recognizes the important fact that the values at risk of the individual categories will not sum to the portfolio's value at risk, unless all the categories are perfectly positively correlated with each other. Thus the independent values at risk could mislead an investor with respect to the sources of a portfolio's exposure to loss. Some investors thus define a risk budget as the sensitivities of a portfolio's value at risk to a small change in the portfolio's exposure to each component.

This definition suffers from a problem of semantics. A budget implies a plan or an action. Given the concept of a risk budget, an investor would plan to convert efficient monetary allocations into efficient value at risk assignments. Yet, the conception of a risk budget as a quantification of a portfolio's sensitivities involves no plan or action. It is merely descriptive. We propose the label, risk attribution, instead of risk budget to characterize this notion.

## RISK ATTRIBUTION

Let us once again consider a portfolio of three assets. In order to calculate its risk attribution, we simply take the partial derivative of its value at risk with respect to each of the assets. We demonstrate this procedure for the portfolio's exposure to asset 1.

We begin by writing the partial derivative of a portfolio's percentage loss in continuous units with respect to exposure to asset 1 as the sum of the derivatives of the two components of a portfolio's percentage loss:

$$\partial L_C/\partial w_1 = \partial \mu_{PC}/\partial w_1 + \partial Z\sigma_{PC}/\partial w_1 \qquad (7)$$

where:

$w_1$ = exposure to asset 1;
$\mu_{PC}$ = expected portfolio return in continuous units;
$Z$ = normal deviate; and
$\sigma_{PC}$ = expected portfolio standard deviation in continuous units.

Next, we define portfolio expected return and standard deviation measured in continuous units as a function of the portfolio's exposure to the component assets:

$$\mu_{PC} = \mu_{1C}w_1 + \mu_{2C}w_2 + \mu_{3C}w_3 \qquad (8)$$

where:

$\mu_{iC}$ = expected return of asset i in continuous units; and:

$$\sigma_{PC} = (\sigma_{1C}^2 w_1^2 + \sigma_{2C}^2 w_2^2 + \sigma_{3C}^2 w_3^2 + 2\rho_{1,2}\sigma_{1C}\, w_1\sigma_{2C}w_2 + 2\rho_{1,3}\sigma_{1C}w_1\sigma_{3C}w_3 + 2\rho_{2,3}\sigma_{2C}w_2\sigma_{3C}w_3)^{1/2} \qquad (9)$$

where:

$\sigma_{iC}$ = standard deviation of asset i in continuous units; and
$\rho_{i,j}$ = correlation of asset i and j in continuous units.

The derivative of portfolio expected return with respect to exposure to asset 1 is straightforward:

$$\partial \mu_{PC}/\partial w_1 = \mu_{1C} \qquad (10)$$

The derivative of $Z\sigma_{PC}$ with respect to exposure to asset 1 is slightly more complicated. We need to invoke the chain rule, by which we first take the partial derivative of $Z\sigma_{PC}$ with respect to portfolio variance, and then multiply it by the partial derivative of portfolio variance with respect to exposure to asset 1:

$$\partial Z\sigma_{PC}/w_1 = \partial Z\sigma_{PC}/\partial \sigma_{PC}^2(\partial \sigma_{PC}^2/\partial w_1) \qquad (11)$$

which equals:

$$\partial Z\sigma_{PC}/w1 = -1/2\ Z(\sigma_{PC}^2)^{-1/2}\ (2\sigma_{1C}^2 w_1 + 2\rho_{1,2}\sigma_{1C}\sigma_{2C}w_2 + 2\rho_{1,3}\sigma_{1C}\sigma_{3C}w_3) \qquad (12)$$

or, more specifically:

$$\partial Z\sigma_{PC}/w_1 = -1/2\ Z(\sigma_{1C}^2 w_1^2 + \sigma_{2C}^2 w_2^2 + \sigma_{3C}^2\ w_3^2 + 2\rho_{1,2}\sigma_{1C}w_1\sigma_{2C}w_2 + 2\rho_{1,3}\sigma_{1C}w_1\sigma_{3C}w_3 + 2\rho_{2,3}\sigma_{2C}w_2\sigma_{3C}w_3) \times (2\sigma_{1C}^2 w_1 + 2\rho_{1,2}\sigma_{1C}\sigma_{2C}w_2 + 2\rho_{1,3}\sigma_{1C}\sigma_{3C}w3) \qquad (13)$$

The final step is to convert this sensitivity of percentage loss measured in continuous units to the sensitivity of the portfolio's value at risk measured in monetary units:

$$\partial\, VaR/w_1 = -e^{(\partial LC/\partial w)} - 1 \qquad (14)$$

Exhibit 2 shows the risk attribution of the same portfolio presented in Exhibit 1.

There are two interesting features of risk attribution.

1. It is mathematically impossible to partition a portfolio's total value at risk into the fractions associated with the individual components, because value at risk is a function of standard deviation, which is not additive, and because it is impossible to disentangle the interaction effect of the various components.
2. The ranking of the portfolio's value at risk sensitivities will not necessarily match the ranking of the portfolio's percentage exposures or the ranking of the individual components' values at risk. Note, for example, that asset 1 has the smallest percentage allocation, yet it has the greatest impact on the portfolio's value at risk. Also, note that although asset 2's value at risk is lower than asset 3's, an additional $1 allocation to asset 2 increases portfolio value at risk by $0.0866, while the same increase in allocation to asset 3 raises portfolio value at risk by less than half as much, $0.0387. The reason is that even though asset 2 has a lower individual value at risk, this advantage is more than offset by its higher correlation with the portfolio.

**EXHIBIT 2**
**RISK ATTRIBUTION**

| Asset | Percentage Allocation (%) | Value at Risk (per $100) | VaR Sensitivity |
|---|---|---|---|
| 1 | 20.70 | 4.9567 | 0.1846 |
| 2 | 30.73 | 5.5429 | 0.0866 |
| 3 | 48.57 | 4.6663 | 0.0387 |
| Portfolio | 100.00 | 7.1453 | |

This risk attribution conveys a very important message. If we wish to limit the potential loss of our portfolio, we should focus not on its largest holding, or on its most volatile asset, and not even on the asset with the greatest value at risk, but instead on the asset to which the portfolio's value at risk is most sensitive.

## SUMMARY

Risk budgets are gaining widespread attention among institutional investors, but they are sometimes applied inappropriately or they are mislabeled.

Investors who regard a risk budget as a plan for assigning value at risk to various categories independently of mean-variance optimization will almost certainly produce an inefficient result.

Investors who regard a risk budget as a process by which to choose a portfolio by minimizing value at risk implicitly care only about a single outcome. They are indifferent to the extent to which a portfolio might generate a cushion with respect to value at risk, and they are no more averse to losses far greater than value at risk than they are to a loss equal to value at risk.

Portfolio choice, instead, should be based on consideration of portfolios all along the efficient frontier and their entire probability distributions, because this process produces efficient value at risk results, and it implicitly considers the pleasure or disutility associated with all possible outcomes, including both gains and losses.

The proper conception of a risk budget is the conversion of optimal percentage allocations from mean-variance optimization into value at risk assignments. This conversion preserves the economic exposures of the portfolio, and it enables investors to achieve these exposures with much greater latitude.

Some investors regard a risk budget as a description of the sensitivity of a portfolio's value at risk to small changes in asset exposures. While this information is valuable, it should properly be called risk attribution because it involves no plan or action. It is merely descriptive.

## ENDNOTES

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[1]Returns are lognormally distributed owing to the effect of compounding. If periodic returns are lognormally distributed, it follows that continuous returns are normally distributed. Thus, we use the normal distribution to estimate probability of loss, but we must do so in continuous return units and then convert them to periodic returns.

[2]Value at risk can be computed for any probability, although 5.00% seems to be the most common choice.

[3]Bootstrapping means that the empirical distribution rather than a theoretical distribution is sampled randomly with replacement.

[4]The Basle Committee comprises senior officials from the central banks of the G-10 countries (Belgium, Canada, France, Germany, Italy, Japan, the Netherlands, Sweden, the U.K., and the U.S.), plus Luxembourg and Switzerland.

## REFERENCE

Jorion, Philippe. *Value at Risk: The New Benchmark for Controlling Market Risk.* New York: McGraw-Hill, 1997.